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# Literal Approximations to Aircraft Dynamic Modes

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**Derivation of literal approximations to the aircraft dynamic modes remains one of the central problems in the theory of aircraft flight dynamics, and one that has never been satisfactorily resolved. Although approximations to the fast modes (short period, roll) are generally good, the slow modes (phugoid, dutch roll, spiral) seem to have several competing approximations, none of which are uniformly satisfactory. This paper examines the fundamental assumptions behind the traditional approach to the derivation of literal approximations. A major flaw in all previous derivations resulting in incorrect slow-mode approximations is uncovered. In following a formal procedure, improved literal approximations are derived for the slow modes. In this process, a new condition for onset of directional departure is obtained, which is related to the dutch-roll frequency. Predictions from the literal approximations derived in this paper are compared with actual numerical values for an example aircraft to illustrate conditions under which the approximations work well, and to point out the fundamental limitations of the literal approximations derived by the traditional approach.**

## Nomenclature

| | | |
|---|---|---|
| $b$ | = | aircraft wingspan |
| $C_D, C_L, C_Y$ | = | drag, lift, and sideforce coefficients, respectively |
| $C_l, C_m, C_n$ | = | roll, pitch, and yaw coefficients, respectively |
| $c$ | = | mean aerodynamic chord |
| $g$ | = | acceleration due to gravity |
| $I_x, I_y, I_z$ | = | roll, pitch, and yaw moments of inertia, respectively |
| $L, M, N$ | = | roll, pitch, and yaw moments, respectively |
| $m$ | = | aircraft mass |
| $p, q, r$ | = | body-axis roll, pitch, and yaw rates, respectively |
| $\bar{q}$ | = | dynamic pressure |
| $S$ | = | reference area |
| $T_m$ | = | maximum thrust |
| $V$ | = | aircraft velocity |
| $X, Y, Z$ | = | force components along aircraft $x$, $y$, $z$ axes |
| $X_u, X_q, \ldots$ | = | dimensional stability derivatives with respect to $u, q, \ldots$ |
| $\alpha, \beta$ | = | angles of attack and sideslip, respectively |
| $\gamma$ | = | flight-path angle |
| $\Delta V, \Delta\alpha, \ldots$ | = | perturbations in state variables $V, \alpha, \ldots$ |
| $\delta e, \delta a, \delta r$ | = | elevator, aileron, and rudder deflections, respectively |
| $\zeta$ | = | modal damping |
| $\eta$ | = | throttle, as fraction of maximum thrust |
| $\mu$ | = | wind-axis roll orientation angle |
| $\rho$ | = | air density |
| $\phi, \theta$ | = | body-axis roll and pitch Euler angles, respectively |
| $\omega$ | = | modal frequency |

*Subscripts*

| | | |
|---|---|---|
| DR | = | dutch roll |
| lat | = | pertaining to lateral dynamics |
| lon | = | pertaining to longitudinal dynamics |
| P | = | phugoid |
| R | = | roll |
| S | = | spiral |
| SP | = | short period |
| $s$ | = | static residual value |
| 0 | = | trim, or equilibrium, value |

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## I. Introduction

THE flight dynamics of rigid aircraft are modeled by a set of eight first-order differential equations, presented in full in the Appendix, of the form

$$\dot{x} = f(x, u) \tag{1}$$

where the states $x$ and controls $u$ are as follows:

$$x = [V, \alpha, \beta, p, q, r, \phi, \theta], \qquad u = [\eta, \delta e, \delta a, \delta r]$$

Traditionally, a study of flight dynamics proceeds by considering the aircraft to be flying in a straight and level flight trim. Then, the two fundamental problems of flight dynamics may be broadly stated as 1) The problem of response to small perturbations in the state variables from the trimmed flight condition, or the problem of stability, and 2) the problem of response to control inputs at the trimmed flight condition. The dynamic response of the aircraft is described in terms of its modal behavior. For straight and level flight trim, the dynamic modes for conventional airplanes decouple into two independent sets: 1) longitudinal modes, called the short period and phugoid, and 2) lateral modes, called the roll, dutch roll, and spiral. Thus, a key step for progress in traditional flight dynamics is to be able to describe these modes. This is best done by developing literal expressions for the individual modes. However, exact literal expressions for these modes turn out to be too complicated to be useful. Flight dynamicists have, therefore, sought to develop useful literal approximations for the aircraft dynamic modes.

Literal approximations to the aircraft dynamic modes have traditionally been derived by considering simplifying assumptions based on the physics of the modes. For example, observations show the short period to be a fast mode with significant variation in angle of attack, and the phugoid appears as a slower mode with oscillation in the aircraft velocity. The short-period approximation, therefore, neglects changes in aircraft velocity, and the phugoid mode is approximated by neglecting dynamics in pitch and assuming no change in angle of attack. Approximations to the lateral modes have also been derived on the basis of such physical arguments and observations. These approximations have been described in several textbooks,[1–3] and have been widely used over the past several decades. Nevertheless, the traditional approach to deriving literal approximations has been recognized to be unsatisfactory on two counts: 1) There is no single unified procedure for deriving the literal approximations. Thus, some of the approximations appear to be ad hoc, and in

**Table 1 Dutch roll approximations**

| Approximation | Variables included (order) | Forces/moments included |
|---|---|---|
| Snaking[4] | $\beta, r$ (2) | Yaw |
| Nelson[2] | $\beta, r$ (2) | Sideforce, yaw |
| Chakravarty[5] | $\beta, r$ (2) | Sideforce, roll, yaw |
| Schmidt[4] | $\beta, p, r$ (3) | Roll, yaw |
| Liebst and Nolan[6] | $\beta, \phi, p$ (3) | Sideforce, roll |

some modes, there is more than one approximation. For example, there are several competing literal approximations for the dutch-roll mode[2,4–6] (see Table 1). 2) Approximations to some of the modes are known to be unsatisfactory, and authors frequently quote them with disclaimers regarding their accuracy. For example, approximations to the phugoid mode often compare poorly with actual aircraft data.

In the absence of a formal procedure for deriving literal approximations, there is no way to decide between competing approximations except to test them against several sets of aircraft data. However, it is extremely difficult to establish the superiority of one approximation over another in this manner, and it is usually found that none of the approximations are uniformly satisfactory. For example, Table 1 lists some of the more prominent dutch-roll approximations obtained in the past and the different variables and various combinations of the force/moment equations that have been considered in these approximations. No single approximation in Table 1 can, however, be stated to be generally better then the others. More puzzling has been the inability to arrive at a satisfactory approximation for the phugoid mode. As pointed out by Pradeep,[7] considering the fact that the approximation for the short period, the other longitudinal mode, is generally found to be quite good, it has been a mystery why the phugoid approximation does not work satisfactorily. Several existing phugoid approximations have been listed by Pradeep[7] and their relative merits compared. Once again, no single phugoid approximation works well across all aircraft data.

Improved literal approximations, however, continue to be of interest to flight dynamicists, and, of late, attempts have been made to approach the problem from a mathematical viewpoint. An improved literal approximation for the phugoid mode was derived by Pradeep[7] following a purely mathematical approach, and a different mathematical approach was used by Livneh[8] for the lateral modes. More recent attempts in this direction have been by Phillips for the phugoid[9] and for the dutch-roll modes.[10] Although these studies claim improved literal approximations that provide better accuracy, this is obtained at the cost of more complicated expressions and a lack of physical understanding. Also, some of these approximations are based on a particular set of aircraft data, and their validity for airplanes outside the assumed set is not immediately obvious. In any case, it is clear that no study based on a purely mathematical approach is capable of explaining the drawbacks in the existing literal approximations.

In this paper, we follow, for the first time, a formal procedure for deriving literal approximations to the aircraft dynamic modes. The derivation is based on a reformulation of the small-perturbation aircraft dynamic equations about a straight and level flight trim in second-order form. Using this procedure, it is possible to formally derive a single literal approximation to each mode, thus resolving the problem of multiplicity of approximations. We also point out a major flaw in previous textbook derivations that was responsible for the poor approximations to the phugoid and dutch-roll modes (in general, the slow modes). Improved phugoid and dutch-roll approximations are obtained when this flaw is taken care of. Satisfactory existing literal approximations (generally, for the fast modes) are recovered in our procedure. Incidentally, in this process, an improved criterion for directional departure, linked to the dutch-roll frequency, is obtained. Finally, as an illustration, predictions from the literal approximations are compared with actual numerical values for an example aircraft. It must be emphasized that the results derived in this paper are of a fundamental nature, and are not particular to any class of airplanes or any particular set of aircraft data.

## II. Linearized Equations in Second-Order Form

Following the standard practice, the trim state is chosen to be a straight and level flight condition. Then, the trim values of the states and controls are as follows:

$$x = [V_0, \alpha_0, \beta_0 = 0, p_0 = 0, q_0 = 0, r_0 = 0, \phi_0 = 0, \theta_0]$$

$$u = [\eta_0, \delta e_0, \delta a_0 = 0, \delta r_0 = 0]$$

with the added condition on the flight path angle, $\gamma_0 = 0$, which at trim is the same as $\theta_0 = \alpha_0$. The linearized equations for the aircraft dynamics are obtained by considering small perturbations in the states with the control parameters fixed at their trim values. Writing the state variables as a sum of the trim value and a small perturbation, for example, $V = V_0 + \Delta V$ in Eq. (1), the linearized small-perturbation equations in the eight state variables can be obtained. It is well known[1–3] that the eight linearized dynamic equations naturally decouple into two sets, longitudinal and lateral, with the corresponding perturbed state variables as follows:

$$\Delta x_{\text{lon}} = [\Delta V, \Delta\alpha, \Delta q, \Delta\theta], \qquad \Delta x_{\text{lat}} = [\Delta\beta, \Delta p, \Delta r, \Delta\phi]$$

Using this notation, the small-perturbation equations for the longitudinal dynamics are a set of four linearized equations of the form $\Delta\dot{x}_{\text{lon}} = A_{\text{lon}}\Delta x_{\text{lon}}$, written in full as

$$\Delta\dot{V} = X_u\Delta V + X_\alpha\Delta\alpha - g\Delta\theta$$

$$\Delta\dot{\alpha} = (Z_u/V_0)\Delta V + (Z_\alpha/V_0)\Delta\alpha + \Delta q$$

$$\Delta\dot{q} = M_u\Delta V + M_\alpha\Delta\alpha + M_q\Delta q, \qquad \Delta\dot{\theta} = \Delta q \tag{2}$$

and the linearized equations for the lateral dynamics in the perturbed state variables are of the form $\Delta\dot{x}_{\text{lat}} = A_{\text{lat}}\Delta x_{\text{lat}}$ and appear in full as follows:

$$\Delta\dot{\beta} = (Y_\beta/V_0)\Delta\beta + (g\cos\theta_0/V_0)\Delta\phi + \sin\theta_0\Delta p - \cos\theta_0\Delta r$$

$$\Delta\dot{p} = L_\beta\Delta\beta + L_p\Delta p + L_r\Delta r$$

$$\Delta\dot{r} = N_\beta\Delta\beta + N_p\Delta p + N_r\Delta r, \qquad \Delta\dot{\phi} = \Delta p + \tan\theta_0\Delta r \tag{3}$$

where the dimensional stability derivatives are defined in the standard manner,[1–3] and, for convenience, rate derivatives for the force equations and derivatives with respect to $\dot{\alpha}$ are not considered.

In a significant departure from the standard practice, the linearized first-order Eqs. (2) and (3) are recast in the second-order form as follows:

$$\Delta\ddot{y}_{\text{lon}} + C_{\text{lon}}\Delta\dot{y}_{\text{lon}} + K_{\text{lon}}\Delta y_{\text{lon}} = 0$$

$$\Delta\ddot{y}_{\text{lat}} + C_{\text{lat}}\Delta\dot{y}_{\text{lat}} + K_{\text{lat}}\Delta y_{\text{lat}} = 0$$

with the variables $\Delta y$ for the second-order form taken as

$$\Delta y_{\text{lon}} = [\Delta V, \Delta\alpha], \qquad \Delta y_{\text{lat}} = [\Delta\beta, \Delta\mu]$$

The roll orientation angle in wind axes, $\Delta\mu$, is chosen in preference to $\Delta\phi$ to avoid a $\cos\theta_0$ denominator term. The two are related by the following expression:

$$\Delta\mu = \cos\theta_0\Delta\phi$$

To derive the second-order equation in $\Delta V$, for example, one takes Eq. (2) for $\Delta\dot{V}$, differentiates once with respect to time, and replaces all terms in $\Delta q$, $\Delta\theta$ and their derivatives in terms of $\Delta V$, $\Delta\alpha$ and derivatives by using the other equations in Eq. (2). The resulting longitudinal equations in second-order form appear as follows:

$$\Delta\ddot{V} - X_u\Delta\dot{V} - (X_\alpha - g)\Delta\dot{\alpha} - (gZ_u/V_0)\Delta V - (gZ_\alpha/V_0)\Delta\alpha = 0 \tag{4}$$

$$\Delta\ddot{\alpha} - (Z_u/V_0)\Delta\dot{V} - (M_q + Z_\alpha/V_0)\Delta\dot{\alpha} - [M_u - M_q(Z_u/V_0)]\Delta V - [M_\alpha - M_q(Z_\alpha/V_0)]\Delta\alpha = 0 \tag{5}$$

The second-order equations for the lateral dynamics can be similarly derived as follows:

$$\ddot{\Delta}\beta + [-(N_r' \cos\alpha_0 - N_p' \sin\alpha_0) - Y_\beta/V_0]\dot{\Delta}\beta + [(N_r' \sin\alpha_0 + N_p' \cos\alpha_0) - g/V_0]\dot{\Delta}\mu + [N_\beta' + (Y_\beta/V_0)(N_r' \cos\alpha_0 - N_p' \sin\alpha_0)]\Delta\beta + [(g/V_0)(N_r' \cos\alpha_0 - N_p' \sin\alpha_0)]\Delta\mu = 0 \tag{6}$$

$$\ddot{\Delta}\mu + [(L_r' \cos\alpha_0 - L_p' \sin\alpha_0)]\dot{\Delta}\beta - [(L_r' \sin\alpha_0 + L_p' \cos\alpha_0)]\dot{\Delta}\mu - [L_\beta' + (Y_\beta/V_0)(L_r' \cos\alpha_0 - L_p' \sin\alpha_0)]\Delta\beta - [(g/V_0)(L_r' \cos\alpha_0 - L_p' \sin\alpha_0)]\Delta\mu = 0 \tag{7}$$

where

$$N_\beta' = N_\beta \cos\alpha_0 - L_\beta \sin\alpha_0, \qquad L_\beta' = L_\beta \cos\alpha_0 + N_\beta \sin\alpha_0$$

$$N_p' = N_p \cos\alpha_0 - L_p \sin\alpha_0, \qquad L_p' = L_p \cos\alpha_0 + N_p \sin\alpha_0$$

$$N_r' = N_r \cos\alpha_0 - L_r \sin\alpha_0, \qquad L_r' = L_r \cos\alpha_0 + N_r \sin\alpha_0$$

The first- and second-order forms are entirely equivalent. However, the second-order form presents us with a significant advantage in that it is possible to explicitly separate the terms in each of the $4 \times 4$ $A_{\mathrm{lon}}$ and $A_{\mathrm{lat}}$ matrices into damping and stiffness terms in the respective $C$ and $K$ matrices. This often provides useful physical insight into the contribution from a particular term. It is also often easier to manipulate the $2 \times 2$ $C$ and $K$ matrices as against the $4 \times 4$ $A$ matrix. The linearized equations in second-order form are next used to derive criteria for longitudinal and lateral static instability, partly to demonstrate the advantage of writing the equations in second-order form, and partly in anticipation of their use in the derivation of the literal approximations.

## III. Static Instability Criteria

The condition for onset of static instability as given by Routh's criterion is that the constant term in the characteristic equation must be zero. Using the linearized equations in second-order form, the characteristic equation can be written as

$$|\lambda^2 I + \lambda C + K| = 0$$

where $I$ is the $2 \times 2$ identity matrix. It is easy to see that the zero constant term condition corresponds to $|K| = 0$. The matrices $K_{\mathrm{lon}}$ and $K_{\mathrm{lat}}$ can be written as follows, by inspection of Eqs. (4–7):

$$K_{\mathrm{lon}} = \begin{bmatrix} -gZ_u/V_0 & -gZ_\alpha/V_0 \\ -(M_u - M_q Z_u/V_0) & -(M_\alpha - M_q Z_\alpha/V_0) \end{bmatrix}$$

$$K_{\mathrm{lat}} = \begin{bmatrix} N_\beta' + (Y_\beta/V_0)(N_r' \cos\alpha_0 - N_p' \sin\alpha_0) & (g/V_0)(N_r' \cos\alpha_0 - N_p' \sin\alpha_0) \\ -L_\beta' - (Y_\beta/V_0)(L_r' \cos\alpha_0 - L_p' \sin\alpha_0) & -(g/V_0)(L_r' \cos\alpha_0 - L_p' \sin\alpha_0) \end{bmatrix}$$

The condition for onset of longitudinal static instability is given by $|K_{\mathrm{lon}}| = 0$. The longitudinal static instability condition ($S_{\mathrm{lon}}$):

$$(g/V_0)(Z_u M_\alpha - Z_\alpha M_u) = 0$$

agrees with the criterion cited by Etkin and Reid.[1]

The condition for onset of the first lateral static instability is given by $|K_{\mathrm{lat}}| = 0$. The first lateral static instability condition ($S^1_{\mathrm{lat}}$):

$$(g/V_0)[(L_\beta' N_r' - N_\beta' L_r') \cos\alpha_0 - (L_\beta' N_p' - N_\beta' L_p') \sin\alpha_0] = 0 \tag{8}$$

is the same as the criterion stated by Etkin and Reid.[1] The condition $S^1_{\mathrm{lat}}$ usually corresponds to onset of a divergent spiral mode.

Often a mildly divergent spiral mode is tolerated, and what is really of interest is the next onset of static instability in the lateral motion, after neglecting the spiral mode. To derive a condition for this, we assume that Eq. (8) is satisfied. On factoring out the zero root corresponding to $S^1_{\mathrm{lat}}$, the characteristic equation for the lateral dynamics is effectively of third order. The condition for the next zero root is given by the constant term of this third-order equation, which is the same as the linear term of the original fourth-order characteristic equation, being zero. This gives, after some manipulation, the following condition for onset of the second static instability in the lateral motion ($S^2_{\mathrm{lat}}$):

$$(N_\beta' L_p' - L_\beta' N_p') \cos\alpha_0 - (L_\beta' N_r' - N_\beta' L_r') \sin\alpha_0 + gL_\beta'/V_0 + (Y_\beta/V_0)(N_r' L_p' - L_r' N_p') = 0 \tag{9}$$

This is a new result, although a similar expression, but with the gravity term neglected, has been derived by Lutze et al.[11] The static instability predicted by Eq. (9) is usually described as directional departure. It requires several simplifying assumptions to reduce condition $S^2_{\mathrm{lat}}$ to the popular directional departure criterion, $N_\beta' = N_\beta \cos\alpha_0 - L_\beta \sin\alpha_0$, also called $N_\beta$(dyn).[11] However, these assumptions are often not physically justifiable, and, hence, $N_\beta$(dyn) is not always a satisfactory criterion for the onset of directional departure.

## IV. Literal Approximations: Longitudinal Modes

Equations (4–7) in second-order form are the exact linearized equations for small-perturbation motion around the chosen straight and level flight trim. Beginning with these equations, literal approximations to the various modes can be derived by following a formal procedure that we will show. It is important to realize that the traditional approach to derivation of literal approximations is based on the premise that the eigenvalues of the different modes in each case, longitudinal and lateral, are well separated in the complex plane, which allows one to distinguish between fast and slow modes. Clearly, the literal approximations will not be effective if this assumption is violated, as we show, later, in the illustrative calculations.

The longitudinal dynamics, Eqs. (4) and (5), represent two equations of second order in the variables $\Delta V$ and $\Delta\alpha$. For conventional airplanes, the four longitudinal mode eigenvalues are found to consist of two complex conjugate pairs representing two oscillatory modes, with the faster mode called the short period and the slower one the phugoid or long period. Observations show that the main variable in the short period is the angle of attack, $\Delta\alpha$, and the phugoid mode mainly consists of variation in the velocity, $\Delta V$. Thus, the first step in the derivation of the literal approximations is to associate the variable $\Delta\alpha$ with the short-period motion, and $\Delta V$ with the phugoid mode.

### A. Short-Period Mode

As a result of the first step, the short-period motion is considered to be described by Eq. (5) for the perturbed angle-of-attack dynamics. It is next assumed that because the velocity varies per the slower phugoid mode, the rate of change of the perturbed velocity $\dot{\Delta}V$ during the short-period motion is negligible, that is, $\dot{\Delta}V \approx 0$. With this assumption, Eq. (5) for the short-period dynamics may be written as

$$\ddot{\Delta}\alpha - (M_q + Z_\alpha/V_0)\dot{\Delta}\alpha - [M_\alpha - M_q(Z_\alpha/V_0)]\Delta\alpha = [M_u - M_q(Z_u/V_0)]\Delta V \tag{10}$$

The zero perturbed velocity rate assumption is good when the short-period and phugoid eigenvalues are well separated in the complex plane, as is usually the case. However, if these two pairs of eigenvalues approach each other in the complex plane, the assumption of zero rate of change of perturbed velocity will progressively worsen, and so will the accuracy of the literal approximation.

The term in $\Delta V$ has been retained and transferred to the right-hand side of Eq. (10) because the perturbed velocity $\Delta V$ depends on the initial condition and is, in general, not zero. However, because the rate of change of $\Delta V$ has been considered to be negligible, the term on the right-hand side of Eq. (10) essentially acts as a static forcing term and does not affect the fast dynamics in angle of attack. The short-period frequency and damping may therefore be read from the left hand side of Eq. (10) as follows:

$$\omega_{SP}^2 = -[M_\alpha - M_q(Z_\alpha/V_0)], \qquad 2\zeta_{SP}\omega_{SP} = -(M_q + Z_\alpha/V_0)$$

This is the standard approximation and is generally found to be quite accurate.[1–3] However, because of the presence of the forcing term on the right-hand side, the perturbed angle of attack does not tend to zero, but to a static value:

$$\Delta\alpha_s = -\Delta V[M_u - M_q(Z_u/V_0)]/[M_\alpha - M_q(Z_\alpha/V_0)] \tag{11}$$

Thus, at the end of the short-period mode, there is a static residual value of the perturbed angle of attack, $\Delta\alpha_s$, whose variation is then governed by the slower rate of evolution of the phugoid dynamics in $\Delta V$. Previous literal approximations to the longitudinal modes effectively considered the perturbed velocity $\Delta V$ to be zero, thus overlooking the presence of the static residual in the fast mode variable.[1–3] Although the static residual does not alter the short-period dynamics, the presence of the static residual will prove to be an important factor in arriving at a literal approximation for the phugoid mode. This explains why previous literal approximations for the fast modes (short-period, roll) were satisfactory, but the approximations for the slower modes (phugoid, dutch-roll) were poor. The neglect of the fast-mode static residual while deriving the approximations for the slower modes has been a major flaw in all previous derivations of literal approximations following the traditional approach.

### B. Phugoid Mode

The first step in the derivation of the literal approximations associated the phugoid mode with the variable $\Delta V$. The phugoid motion is therefore described by Eq. (4) for the perturbed velocity. However, Eq. (4) also contains terms in $\Delta\alpha$ and $\Delta\dot{\alpha}$, which must be correctly interpreted as representing the static residual value of the perturbed angle of attack and its rate, whose variation is now governed by the slower phugoid time period. The relation between the static residual $\Delta\alpha_s$ and the perturbed velocity is given by Eq. (11), and the rate of change of the static residual $\Delta\dot{\alpha}_s$ in terms of the rate of change of the perturbed velocity $\Delta\dot{V}$ is obtained by differentiating both sides of Eq. (11) with respect to time. When these expressions are used in place of the perturbed angle of attack and its rate in Eq. (4), the equation for the phugoid dynamics appears as

$$\Delta\ddot{V} + \left[-X_u + (X_\alpha - g)\frac{M_u V_0 - M_q Z_u}{M_\alpha V_0 - M_q Z_\alpha}\right]\Delta\dot{V} + \frac{g(M_u Z_\alpha - M_\alpha Z_u)}{M_\alpha V_0 - M_q Z_\alpha}\Delta V = 0 \tag{12}$$

The new literal approximations for the phugoid frequency and damping are obtained from the left-hand side of Eq. (12) as follows:

$$\omega_P^2 = \frac{g(M_u Z_\alpha - M_\alpha Z_u)}{M_\alpha V_0 - M_q Z_\alpha}$$

$$2\zeta_P\omega_P = -X_u + (X_\alpha - g)\frac{M_u V_0 - M_q Z_u}{M_\alpha V_0 - M_q Z_\alpha}$$

Coincidentally, this phugoid frequency approximation matches exactly with that derived by Pradeep[7] by a different approach, and the phugoid damping derived here is a simplified version of that of Pradeep.

It is interesting to observe that previous literal approximations that did not consider the static residual value of perturbed angle of attack, but assumed $\Delta\alpha$ and $\Delta\dot{\alpha}$ to be zero during the phugoid mode, essentially ended up with the following equation for the phugoid dynamics from Eq. (4):

$$\Delta\ddot{V} - X_u\Delta\dot{V} - (gZ_u/V_0)\Delta V = 0$$

from which the phugoid frequency and damping could be inferred to be $\omega_P^2 = -gZ_u/V_0$ and $2\zeta_P\omega_P = -X_u$, which are usually found to be poor approximations.

## V. Literal Approximations: Lateral Modes

The lateral dynamics, Eqs. (6) and (7), represent two equations of second order in the variables $\Delta\beta$ and $\Delta\mu$. For conventional airplanes, they provide four lateral-mode eigenvalues that are usually found to consist of one complex conjugate pair representing an oscillatory mode and two real eigenvalues. The larger (in magnitude) of the two real eigenvalues represents a fast mode with predominantly roll motion, called the roll mode. The oscillatory mode is called the dutch roll, and the slower mode represented by the other real eigenvalue is called the spiral. The lateral dynamics are complicated by the interaction between the dutch-roll and spiral modes, both of which involve roll, yaw, and sideslipping motion.

To derive literal approximations for the lateral modes along similar lines as for the longitudinal dynamics, it becomes necessary to consider two of these modes at a time. The roll mode is generally the fastest lateral mode, and the spiral is usually the slowest. Therefore, the first step is to drop the spiral mode and consider the dynamics in the roll and dutch-roll modes. To this end, we first assume that gravity can be neglected by taking the limit as $g/V_0$ goes to zero. Neglecting gravity creates a column of zeros in the $A_{lat}$ matrix [see Eq. (3)], implying the presence of a zero eigenvalue, which corresponds to the spiral mode. Thus, neglecting gravity eliminates the spiral mode. However, gravity does have a significant influence on the dutch-roll motion, which will be accounted for later.

The linearized lateral dynamics in second-order form, Eqs. (6) and (7), in the limit of zero $g/V_0$, appear as follows:

$$\Delta\ddot{\beta} + [-(N_r'\cos\alpha_0 - N_p'\sin\alpha_0) - Y_\beta/V_0]\Delta\dot{\beta} + [(N_r'\sin\alpha_0 + N_p'\cos\alpha_0)]\Delta\dot{\mu} + [N_\beta' + (Y_\beta/V_0)(N_r'\cos\alpha_0 - N_p'\sin\alpha_0)]\Delta\beta = 0 \tag{13}$$

$$\Delta\ddot{\mu} + [(L_r'\cos\alpha_0 - L_p'\sin\alpha_0)]\Delta\dot{\beta} - [(L_r'\sin\alpha_0 + L_p'\cos\alpha_0)]\Delta\dot{\mu} - [L_\beta' + (Y_\beta/V_0)(L_r'\cos\alpha_0 - L_p'\sin\alpha_0)]\Delta\beta = 0 \tag{14}$$

where the terms in $\Delta\mu$ may be seen to have dropped out as a result of the vanishing gravity assumption. Thus, Eqs. (13) and (14) represent two equations of second order in $\Delta\beta$ and first order in $\Delta\dot{\mu}$ for the roll-mode and dutch-roll dynamics. Following a similar procedure as for the longitudinal modes, the next step in the derivation of the literal approximation for the lateral modes is to associate the perturbed roll-rate variable $\Delta\dot{\mu}$ with the roll mode, and the perturbed sideslip variable $\Delta\beta$ with the dutch-roll mode.

### A. Roll Mode

The roll mode is represented by the first-order dynamics of Eq. (14) in the perturbed roll rate $\Delta\dot{\mu}$. It is next assumed that because the sideslip varies per the slower dutch-roll mode, the rate of change of the perturbed sideslip $\Delta\dot{\beta}$ during the roll mode is negligible, that is, $\Delta\dot{\beta} \approx 0$. With this assumption, the first-order equation for the roll-mode dynamics may be written as

$$\Delta\ddot{\mu} - (L_r'\sin\alpha_0 + L_p'\cos\alpha_0)\Delta\dot{\mu} = [L_\beta' + (Y_\beta/V_0)(L_r'\cos\alpha_0 - L_p'\sin\alpha_0)]\Delta\beta \tag{15}$$

It is worth pointing out that the zero sideslip perturbation rate assumption is good provided the roll and dutch-roll eigenvalues are well separated in the complex plane. As before, the perturbation in the slow-mode variable (here, the sideslip) cannot be taken to be zero. Instead, the perturbation in sideslip is transferred to the right-hand side of Eq. (15), where it acts as a static forcing for the roll mode but does not affect the roll-mode dynamics. The rate constant of the roll mode is then given by the roll eigenvalue, which is seen from Eq. (15) to be as follows:

$$\lambda_{\mathrm{R}} = (L'_r \sin\alpha_0 + L'_p \cos\alpha_0)$$

In stability axes, $\alpha_0 = 0$, and $\lambda_{\mathrm{R}} = L_p$, which is the standard expression.[1–3] The perturbed roll rate, therefore, rapidly dies down to a static value given by

$$\Delta\dot{\mu}_s = \frac{-\Delta\beta[L'_\beta + (Y_\beta/V_0)(L'_r\cos\alpha_0 - L'_p\sin\alpha_0)]}{L'_r\sin\alpha_0 + L'_p\cos\alpha_0} \tag{16}$$

Previous literal approximations were guilty of assuming $\Delta\beta$ to be zero during the derivation for the roll mode, due to which the static residual value of the roll rate was overlooked, although the roll-mode eigenvalue was correctly obtained. The presence of the static residual in the roll rate will be seen to have a significant influence on the dutch-roll approximation.

### B. Dutch-Roll Mode

The dutch-roll mode, which has been associated with variations in the sideslip, is represented by Eq. (13) in $\Delta\beta$. When the term in $\Delta\dot{\mu}$ is correctly replaced by the static residual value of the perturbed roll rate from Eq. (16), the equation for the dutch-roll dynamics appears as

$$\Delta\ddot{\beta} + [-(N'_r\cos\alpha_0 - N'_p\sin\alpha_0) - Y_\beta/V_0]\Delta\dot{\beta} + \omega_{\mathrm{DR}}^2\Delta\beta = 0 \tag{17}$$

where

$$\omega_{\mathrm{DR}}^2 = [(N'_\beta L'_p - L'_\beta N'_p)\cos\alpha_0 - (L'_\beta N'_r - N'_\beta L'_r)\sin\alpha_0 + (Y_\beta/V_0)(N'_r L'_p - L'_r N'_p)]/(L'_r\sin\alpha_0 + L'_p\cos\alpha_0) \tag{18}$$

is (the square of) the dutch-roll frequency, and the dutch-roll damping is given by

$$2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} = -(N'_r\cos\alpha_0 - N'_p\sin\alpha_0) - Y_\beta/V_0$$

The numerator of Eq. (18) can be seen to be identical to the expression on the left-hand side of the second lateral static instability criterion $S^2_{\mathrm{lat}}$ derived earlier, when $g/V_0$ is put equal to zero. Recall that $S^2_{\mathrm{lat}}$ corresponded to the condition of a second zero eigenvalue in the lateral dynamics after neglecting the spiral eigenvalue at zero. However, the derivation of Eq. (18) has been carried out by assuming a zero spiral eigenvalue by the vanishing gravity assumption. It is important to realize here that a zero spiral eigenvalue does not imply $g/V_0 = 0$; the converse is actually true. Thus, the correct observation from Eq. (18) and the second lateral static instability criterion $S^2_{\mathrm{lat}}$ is that

$$\omega_{\mathrm{DR}}^2 = \mathrm{LHS}\left[S^2_{\mathrm{lat}}(g/V_0 = 0)\right]\big/\lambda_{\mathrm{R}}$$

where $\omega_{\mathrm{DR}}^2$ is the dutch-roll frequency derived by neglecting the influence of gravity.

This discussion suggests a way to incorporate the effect of gravity in the expression for the dutch-roll frequency. Consider the following factorization of the characteristic polynomial for the lateral dynamics:

$$P_{\mathrm{lat}}(\lambda) = \left(\lambda^2 + 2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}}\lambda + \omega_{\mathrm{DR}}^2\right)(\lambda + \lambda_{\mathrm{R}})(\lambda + \lambda_{\mathrm{S}}) \tag{19}$$

Then, by definition, the left-hand side of $S^2_{\mathrm{lat}}$ is given by the constant term of the polynomial $P_{\mathrm{lat}}(\lambda)/\lambda$ when $\lambda_S = 0$. That is, $\mathrm{LHS}[S^2_{\mathrm{lat}}] = \omega_{\mathrm{DR}}^2\lambda_{\mathrm{R}}$, which can, of course, be used to write the dutch-roll frequency in terms of $S^2_{\mathrm{lat}}$ as

$$\omega_{\mathrm{DR}}^2 = \mathrm{LHS}\left[S^2_{\mathrm{lat}}\right]\big/\lambda_{\mathrm{R}}$$

Then, using the previously derived condition for $S^2_{\mathrm{lat}}$ (with zero spiral eigenvalue, but $g/V_0$ not equal to zero), a complete expression for the dutch-roll frequency including the effect of gravity can be written, giving the new literal approximation for the dutch-roll mode as follows:

$$\omega_{\mathrm{DR}}^2 = [1/(L'_r\sin\alpha_0 + L'_p\cos\alpha_0)][(N'_\beta L'_p - L'_\beta N'_p)\cos\alpha_0 - (L'_\beta N'_r - N'_\beta L'_r)\sin\alpha_0 + gL'_\beta/V_0 + (Y_\beta/V_0)(N'_r L'_p - L'_r N'_p)]$$

$$2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} = -(N'_r\cos\alpha_0 - N'_p\sin\alpha_0) - Y_\beta/V_0$$

In stability axes, the new dutch-roll frequency and damping approximations take the following form:

$$\omega_{\mathrm{DR}}^2 = \left(N_\beta + \frac{Y_\beta}{V_0}N_r\right) - \left(L_\beta + \frac{Y_\beta}{V_0}L_r\right)\frac{N_p}{L_p} + \left(\frac{g}{V_0}\right)\frac{L_\beta}{L_p}$$

$$2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} = -N_r - \frac{Y_\beta}{V_0} \tag{20}$$

Incidentally, overlooking the static residual in the roll rate, and assuming $\Delta\dot{\mu} = 0$ in Eq. (13), would have resulted in the following equation for the dutch-roll dynamics:

$$\Delta\ddot{\beta} + [-(N'_r\cos\alpha_0 - N'_p\sin\alpha_0) - Y_\beta/V_0]\Delta\dot{\beta} + [N'_\beta + (Y_\beta/V_0)(N'_r\cos\alpha_0 - N'_p\sin\alpha_0)]\Delta\beta = 0$$

from which expressions for the dutch-roll frequency and damping could be written as

$$\omega_{\mathrm{DR}}^2 = N'_\beta + (Y_\beta/V_0)(N'_r\cos\alpha_0 - N'_p\sin\alpha_0)$$

$$2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} = -(N'_r\cos\alpha_0 - N'_p\sin\alpha_0) - Y_\beta/V_0$$

which, in stability axes, would reduce to

$$\omega_{\mathrm{DR}}^2 = N_\beta + (Y_\beta/V_0)N_r, \qquad 2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}} = -N_r - Y_\beta/V_0 \tag{21}$$

This is the dutch-roll approximation commonly derived in textbooks.[2,3] The most significant difference between the expressions in Eq. (21) and the new approximations in Eq. (20) for dutch-roll frequency and damping in stability axes is the presence of additional terms involving $L_\beta$ in the new approximation. The dihedral stability derivative $L_\beta$ enters the dutch-roll frequency approximation in Eq. (20) through the second and third terms on the right-hand side and influences the dutch-roll damping $\zeta_{\mathrm{DR}}$ through its presence in $\omega_{\mathrm{DR}}$ in the left-hand side of the equation for $2\zeta_{\mathrm{DR}}\omega_{\mathrm{DR}}$. Although $L_\beta$ has long been recognized as one of the more significant derivatives affecting the dutch-roll dynamics,[2] the present derivation is the first to successfully capture this influence. Notably, Eq. (20) also captures the effect of gravity on the dutch-roll dynamics, which was missing in previous literal approximations.[2,3]

### C. Spiral Mode

Once good approximations are available for the roll and dutch-roll modes, the spiral eigenvalue can be well approximated by referring to the lateral characteristic polynomial $P_{\mathrm{lat}}$ in Eq. (19). The constant term of this polynomial is precisely what was identified with the first lateral static instability criterion $S^1_{\mathrm{lat}}$ in Eq. (8). That is, $\mathrm{LHS}[S^1_{\mathrm{lat}}] = \omega_{\mathrm{DR}}^2\lambda_{\mathrm{R}}\lambda_{\mathrm{S}}$, from which the spiral eigenvalue may be expressed as

$$\lambda_{\mathrm{S}} = \frac{\mathrm{LHS}\left[S^1_{\mathrm{lat}}\right]}{\omega_{\mathrm{DR}}^2\lambda_{\mathrm{R}}}$$

This gives, on using Eq. (8) along with the previously derived approximation for the roll-mode eigenvalue, the new literal approximation to the spiral mode eigenvalue as

$$\lambda_S = \left(\frac{g}{\omega_{DR}^2 V_0}\right)\frac{(L'_\beta N'_r - N'_\beta L'_r)\cos\alpha_0 - (L'_\beta N'_p - N'_\beta L'_p)\sin\alpha_0}{(L'_r\sin\alpha_0 + L'_p\cos\alpha_0)}$$

where $\omega_{DR}^2$ has not been replaced with the approximation derived earlier for typographical reasons.

The spiral approximation in stability axes, where $\alpha_0 = 0$, then appears as

$$\lambda_S = \frac{(g/V_0)(L_\beta N_r - N_\beta L_r)}{[(g/V_0)L_\beta] + [N_\beta L_p - L_\beta N_p] + [(Y_\beta/V_0)(N_r L_p - L_r N_p)]} \tag{22}$$

When the terms in the second and third pairs of brackets in the denominator are neglected, Eq. (22) for $\lambda_S$ reduces to that given by Nelson,[2] which is, however, found to be a poor approximation to the actual spiral eigenvalue. In general, it is necessary to retain the terms in all three pairs of brackets in the denominator of the spiral approximation in Eq. (22). However, a quick numerical survey of several sets of data provided by Schmidt[4] suggests that the term in the second brackets is the dominant one, and if a simplified version of Eq. (22) is at all desired, then the first and third bracketed terms in the denominator may be neglected to give the following simplified formula for the spiral eigenvalue in stability axes:

$$\lambda_S = \frac{(g/V_0)(L_\beta N_r - N_\beta L_r)}{(N_\beta L_p - L_\beta N_p)}$$

The spiral approximation derived does indeed show a zero spiral eigenvalue in the limit of zero $g/V_0$, as was inferred at the beginning of this section. It is astonishing that previous approximations for the spiral eigenvalue (e.g., see Ref. 3) did not satisfy this obvious requirement, and not suprisingly, performed quite poorly.

## VI. Numerical Illustration

As an illustration, predictions from the literal approximations derived in this paper are now compared with actual numerical values for the aircraft data in the appendix. Eigenvalues for the linearized dynamics around a straight and level flight condition have been numerically computed from these data for trim angles of attack varying between −5 and 25 deg. The longitudinal-mode eigenvalues so computed have been plotted in Fig. 1, and Fig. 2 shows the lateral-mode eigenvalues. The aircraft data in the appendix prove to be useful because they contain both ranges of trim angle of attack over which the approximations work well, and ranges over which the approximations deviate from the actual values. This helps highlight the fundamental assumptions behind the traditional approach to deriving literal approximations, and to point out circumstances in which they may be expected to fail.

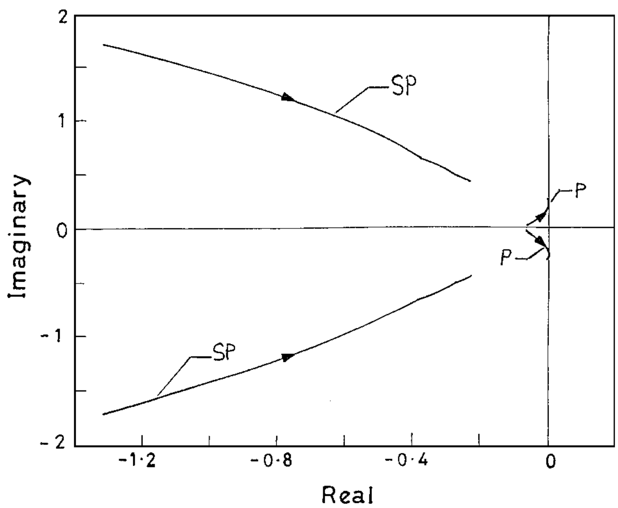


**Fig. 1 Locus of longitudinal eigenvalues with trim angle of attack varying from − 5 to 25 deg (SP, short period; P, phugoid; and arrows indicate direction of increasing trim angle of attack).**

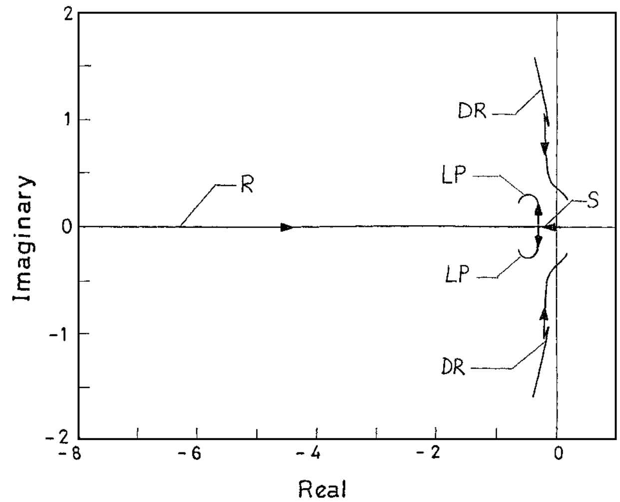


**Fig. 2 Locus of lateral eigenvalues with trim angle of attack varying from − 5 to 25 deg (DR, dutch roll; R, roll; S, spiral; LP, lateral phugoid; and arrows indicate direction of increasing trim angle of attack).**

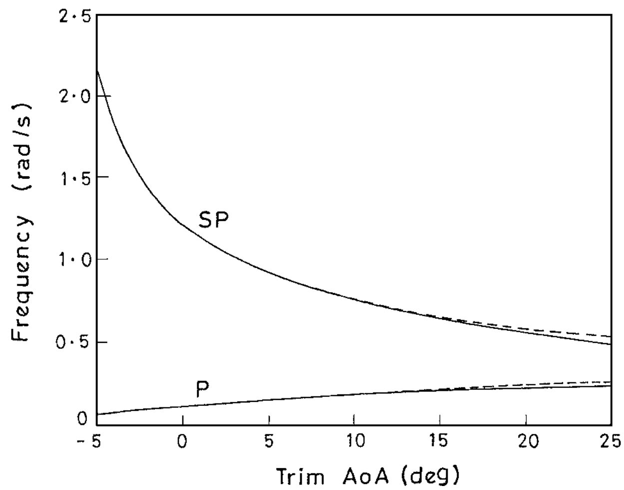


**Fig. 3 Comparison of literal approximation with actual values: longitudinal-mode frequencies (——, actual value; - - -, literal approximation; SP, short period; and P, phugoid).**

The short-period and phugoid frequencies predicted by the literal approximations in this paper are compared with the actual values in Fig. 3. The predicted values literally lie on top of the actual values over most of the trim angle-of-attack range. The approximations to the short-period and phugoid dampings are compared to the actual values in Fig. 4. The approximations match the actual values quite well at low-to moderate-trim angles of attack; the deviation from the actual values at large-trim angles of attack is discussed later. Interestingly, the phugoid damping approximation derived by Pradeep[7] seems to perform quite well in Fig. 4, even though the data in the appendix did not figure in the set of aircraft data he considered. The roll and spiral eigenvalues predicted by the literal approximations derived in this paper are compared to the actual eigenvalues in Fig. 5. The approximations are found to be excellent over most of the trim angle-of-attack range. A comparison of the dutch-roll frequency approximation with the actual values is shown in Fig. 6, which also contains a plot of the existing dutch-roll frequency approximation.[2,3] Clearly, the new approximation seems to predict the actual values well, except at large values of trim angle of attack, and is definitely superior to the existing approximation shown in Fig. 6. The new dutch-roll damping approximation is compared to the actual values in Fig. 7, which shows an excellent match over most values of the trim angle of attack. The deviation in the predictions made by the literal approximations in Figs. 5 through 7 for the larger values of trim angle of attack is discussed later.

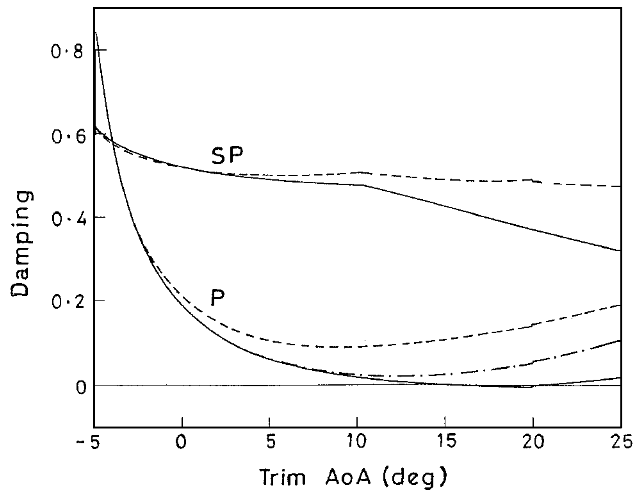


**Fig. 4 Comparison of literal approximation with actual values: longitudinal-mode dampings (——, actual value; - - -, literal approximation; –·–, phugoid damping approximation from Ref. 7; SP, short period; and P, phugoid).**

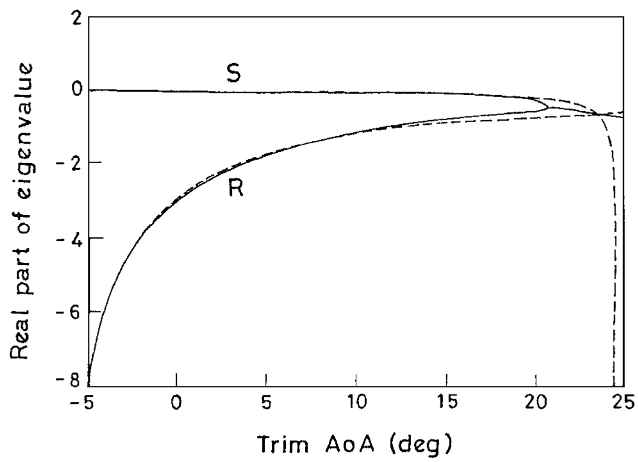


**Fig. 5 Comparison of literal approximation with actual values: roll and spiral eigenvalues (——, actual value; - - -, literal approximation; S, spiral; and R, roll).**

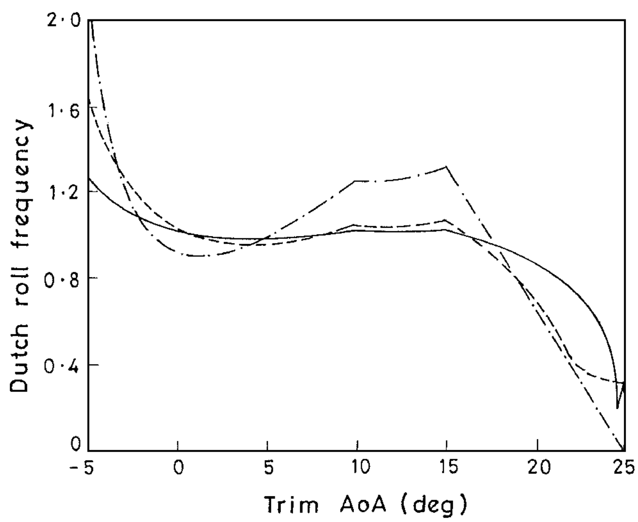


**Fig. 6 Comparison of literal approximation with actual values: dutch-roll frequency (——, actual value; - - -, new literal approximation; and –·–, existing approximation).[2,3]**

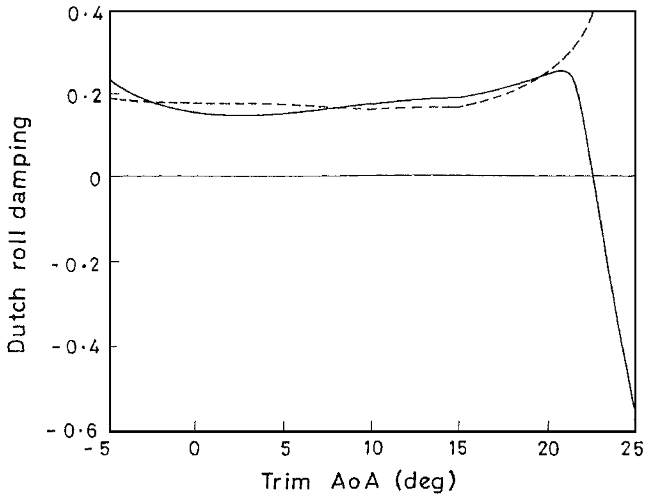


**Fig. 7 Comparison of literal approximation with actual values: dutch-roll damping (——, actual value and - - -, literal approximation).**

It may be noted from Fig. 1 that the short-period and phugoid eigenvalues are well separated for small-trim angles of attack, but for increasing values of trim angle of attack, the short-period eigenvalues approach the phugoid eigenvalues. Thus, for the larger values of trim angle of attack in Fig. 1, the distinction between the fast, short-period mode and the slow, phugoid mode is weaker. Under these circumstances, one may expect the zero perturbed velocity rate assumption in the derivation of the short-period mode to be less appropriate and, therefore, the literal approximations to be less accurate. This is not a failing of the literal approximations per se but rather a consequence of the fact that the traditional approach based on a distinction between fast and slow modes is no longer applicable. Similar is the case of Fig. 2, where the roll and dutch-roll eigenvalues are well separated for small-trim angles of attack, but for larger-trim angles of attack, the roll and spiral eigenvalues combine to form an oscillatory lateral phugoid mode. It may be recalled that first, the lateral approximations were derived by considering the roll and spiral eigenvalues to be real. Second, the roll approximation was based on the roll mode being much faster than the dutch roll, permitting the zero perturbed sideslip rate assumption. That both these assumptions are violated at large-trim angles of attack in Fig. 2 explains the deviation of the literal approximations from the actual values over this range of trim angles of attack. Although this once again points to a fundamental limitation of the traditional approach to deriving literal approximations, the fact must not be missed that the distinction between fast and slow modes remains valid for most conventional airplanes under a wide range of trim flight conditions where, as seen, the literal approximations derived in this paper work well.

## VII. Conclusions

Literal approximations to the aircraft dynamic modes have been formally derived following the traditional approach of distinguishing between fast and slow modes. The importance of the fast-mode static residual in the derivation of the slow-mode approximation has been recognized for the first time, permitting improved approximations for the slow modes to be obtained. In conclusion, it may be appropriate to comment that although literal approximations have limited predictive value in these days of computational ease, they do provide flight dynamicists and aircraft designers with physical insight and the luxury of useful analytical expressions. For these reasons, derivation of literal approximations occupies a central place in the theory of flight dynamics, as evidenced by the elaborate (but unsatisfactory) discussions in textbooks.[1–3] Viewed in this context, the contribution of this paper becomes self-evident, and will, we hope, lead to the existing literal approximations in several otherwise excellent textbooks on flight dynamics being replaced by the derivations in this paper.

## Acknowledgment

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## Appendix: Aircraft Equations and Data

The equations of motion for rigid aircraft dynamics are given by the following set of eight first-order differential equations[12]:

$$\dot{V} = (1/m)\left(T_m \eta \cos\alpha \cos\beta - \tfrac{1}{2} C_D \rho V^2 S - mg \sin\gamma\right) \tag{A1}$$

$$\dot{\alpha} = q - (1/\cos\beta)\left[(p\cos\alpha + r\sin\alpha)\sin\beta + (1/mV)\left(T_m\eta\sin\alpha + \tfrac{1}{2}C_L\rho V^2 S - mg\cos\mu\cos\gamma\right)\right] \tag{A2}$$

$$\dot{\beta} = (1/mV)\left(-T_m\eta\cos\alpha\sin\beta + \tfrac{1}{2}C_Y\rho V^2 S + mg\sin\mu\cos\gamma\right) + (p\sin\alpha - r\cos\alpha) \tag{A3}$$

$$\dot{p} = [(I_y - I_z)/I_x]qr + (1/2I_x)\rho V^2 SbC_l \tag{A4}$$

$$\dot{q} = [(I_z - I_x)/I_y]pr + (1/2I_y)\rho V^2 ScC_m \tag{A5}$$

$$\dot{r} = [(I_x - I_y)/I_z]pq + (1/2I_z)\rho V^2 SbC_n \tag{A6}$$

$$\dot{\phi} = p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \tag{A7}$$

$$\dot{\theta} = q\cos\phi - r\sin\phi \tag{A8}$$

with the wind axis orientation angles, $\mu$ and $\gamma$, defined as follows:

$$\sin\gamma = \cos\alpha\cos\beta\sin\theta - \sin\beta\sin\phi\cos\theta - \sin\alpha\cos\beta\cos\phi\cos\theta \tag{A9}$$

$$\sin\mu\cos\gamma = \sin\theta\cos\alpha\sin\beta + \sin\phi\cos\theta\cos\beta - \sin\alpha\sin\beta\cos\phi\cos\theta \tag{A10}$$

$$\cos\mu\cos\gamma = \sin\theta\sin\alpha + \cos\alpha\cos\phi\cos\theta \tag{A11}$$

The aircraft inertia and geometry data from Ref. 12 are as follows:

Wing area, $S = 400$ ft$^2$, Maximum thrust, $T_m = 11{,}200$ lb

Mean aerodynamic chord, $c = 11.52$ ft

Roll inertia, $I_x = 23{,}000$ slug ft$^2$, Aircraft mass, $m = 1036$ slugs

Pitch inertia, $I_y = 151{,}293$ slug ft$^2$, Wing span, $b = 37.42$ ft

Yaw inertia, $I_z = 169{,}945$ slug ft$^2$

The aerodynamic coefficients in the equations of motion are given in terms of the following polynomial functions, where all angles and control surface deflections are in degrees, and angular rates are in radians per second.[12]

Drag coefficient:

$$C_D = \begin{cases} 0.0013\alpha^2 - 0.00438\alpha + 0.1423, & -5 \le \alpha \le 20 \\ -0.0000348\alpha^2 + 0.0473\alpha - 0.358, & 20 \le \alpha \le 40 \end{cases} \tag{A12}$$

Sideforce coefficient:

$$C_Y = -0.0186\beta + (\delta a/25)(-0.00227\alpha + 0.039) + (\delta r/30)(-0.00265\alpha + 0.141) \tag{A13}$$

Lift coefficient:

$$C_L = \begin{cases} 0.0751\alpha + 0.0144\delta e + 0.732, & -5 \le \alpha \le 10 \\ -0.00148\alpha^2 + 0.106\alpha + 0.0144\delta e + 0.569, & 10 \le \alpha \le 40 \end{cases} \tag{A14}$$

Rolling moment coefficient:

$$C_l = C_l(\alpha,\beta) - 0.0315p + 0.0126r + (\delta a/25)(0.00121\alpha - 0.0628) - (\delta r/30)(0.000351\alpha - 0.0124) \tag{A15}$$

where

$$C_l(\alpha,\beta) = \begin{cases} (-0.00012\alpha - 0.00092)\beta, & -5 \le \alpha \le 15 \\ (0.00022\alpha - 0.006)\beta, & 15 \le \alpha \le 25 \end{cases} \tag{A16}$$

Pitching moment coefficient:

$$C_m = -0.00437\alpha - 0.0196\delta e - 0.123q - 0.1885 \tag{A17}$$

Yawing moment coefficient:

$$C_n = C_n(\alpha,\beta) - 0.0142r + (\delta a/25)(0.000213\alpha + 0.00128) + (\delta r/30)(0.000804\alpha - 0.0474) \tag{A18}$$

where

$$C_n(\alpha,\beta) = \begin{cases} 0.00125\beta, & -5 \le \alpha \le 10 \\ (-0.00022\alpha + 0.00342)\beta, & 10 \le \alpha \le 25 \\ -0.00201\beta, & 25 \le \alpha \le 35 \end{cases} \tag{A19}$$

All data correspond to low Mach number flight at sea-level conditions.

## References

[1] Etkin, B., and Reid, L. D., *Dynamics of Flight: Stability and Control*, Wiley, New York, 1996.

[2] Nelson, R. C., *Flight Stability and Automatic Control*, McGraw-Hill, New York, 1990.

[3] Roskam, J., *Airplane Flight Dynamics and Automatic Flight Controls*, Vol. 1, Roskam Aviation and Engineering Corp., Lawrence, KS, 1979.

[4] Schmidt, L. V., *Introduction to Aircraft Flight Dynamics*, AIAA Education Series, AIAA, New York, 1998.

[5] Chakravarty, M. K., "Prediction, Modeling, and Mechanism of Aircraft Wing Rock," M. S. Thesis, Dept. of Aerospace Engineering, Indian Inst. of Technology, Bombay, India, 1999.

[6] Liebst, B. S., and Nolan, R. C., "Method for Prediction of the Onset of Wing Rock," *Journal of Aircraft*, Vol. 31, No. 6, 1994, pp. 1419–1421.

[7] Pradeep, S., "A Century of Phugoid Approximations," *Aircraft Design*, Vol. 1, Pergamon, Tarrytown, NY, 1998, pp. 89–104.

[8] Livneh, R., "Improved Literal Approximation for Lateral-Directional Dynamics of Rigid Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 4, 1995, pp. 925–927.

[9] Phillips, W. F., "Phugoid Approximation for Conventional Airplanes," *Journal of Aircraft*, Vol. 37, No. 1, 2000, pp. 30–36.

[10] Phillips, W. F., "Improved Closed Form Approximation for Dutch Roll," *Journal of Aircraft*, Vol. 37, No. 3, 2000, pp. 484–490.

[11] Lutze, F. H., Durham, W. C., and Mason, W. H., "Unified Development of Lateral-Directional Departure Criteria," *Journal of Guidance, Control, and Dynamics*, Vol. 19, No. 2, 1996, pp. 489–493.

[12] Fan, Y., Lutze, F. H., and Cliff, E. M., "Time-Optimal Lateral Maneuvers of an Aircraft," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 5, 1995, pp. 1106–1112.